Qatar University
College of Engineering

Numerical Methods -- GENG 300

Part 1

Introduction

true error (E_t) = true value - approximation

true Percent Error = $\left| \frac{E_t}{\text{true value}} \right| * 100\%$

ε_t

* if we can not Get true value Get approximation value

$\varepsilon_a = \left| \frac{\text{Current approx} - \text{Pervious approx}}{\text{Current approx}} \right| * 100\%$

* $|\varepsilon_a| < \varepsilon_s$ Stop

$\Downarrow$

tolerance

* $\varepsilon_s = 0.5 * 10^{2-n}$

$n \Rightarrow$ Significant Figures

Example 4.1

Maclaurin series expansion

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

- Solve for $e^{(0.5)}$ until the approx. error estimate $|\varepsilon_a|$ falls below a prespecified error criterion ε_s conforming to three significant figures:

$n = 3$

$$\varepsilon_s = (0.5 \times 10^{2-n}) = 0.5 \times 10^{2-3}$$

$$= 0.05\ \%$$

* True value :-

$$e^{0.5} = 1.64872$$

1st term :- #

$$e^{0.5} = 1$$

⇒ 2nd term :-

$$e^{0.5} = 1 + x = 1 + 0.5 = 1.5$$

$$\varepsilon_a = \left|\frac{1.5 - 1}{1.5}\right| \times 100\ \% = 33.3\ \% > 0.05\ \%$$

$$\varepsilon_t = \left|\frac{1.648721 - 1.5}{1.648721}\right| \times 100 = 9.02\ \%$$

3rd term :-

$$e^x = 1 + x + \frac{x^2}{2}$$

$$= 1 + 0.5 + \frac{0.5^2}{2} = 1.625$$

$$\varepsilon_a = \left|\frac{1.625 - 1.5}{1.625}\right| \times 100$$

$$= 7.69\ \% > 0.05\ \%$$

4th term :-

Terms	Result	ε_t, %	ε_a, %
1	1	39.3	
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.645833333	0.175	1.27
5	1.648437500	0.0172	0.158
6	1.648697917	0.00142	0.0158

$$|\varepsilon_a| < (\varepsilon_s = 0.05\%)$$

The Taylor Series :-

To Predict a Function value in approximate Method :-

$x_i \Rightarrow$ Given $\Rightarrow$ base point

$F(x_{i+1}) \rightarrow$ Required

$h = x_{i+1} - x_i$

$$F(x_{i+1}) = F(x_i) + F'(x_i)\underbrace{(x_{i+1} - x_i)}_{h} + \frac{F''(x_i)\,h^2}{2!} + \frac{F'''(x_i)\,h^3}{3!} + \frac{F''''(x_i)\,h^4}{4!} + \cdots$$

Zero-order approximation :- 1st term

$$F(x_{i+1}) = F(x_i)$$

1st order approximation :- 2-terms

$$F(x_{i+1}) = F(x_i) + F'(x_i)\,h$$

2nd order approximation :- 3-term

$$F(x_{i+1}) = F(x_i) + F'(x_i)\,h + \frac{F''(x_i)\,h^2}{2!}$$

3rd-order approximation :-

Chapter 4

Approximation of a function with a Taylor series expansion

Example → calculate the approximate value of

of $f(x) = x^3 + 2x^2 + 3$

@ $x_{i+1} = 6$

given $x_i = 4$, True value = 270

Use :-

1. Zero order approximation
2. First order approximation
3. second order approximation

$h = x_{i+1} - x_i$
$= 6 - 4$
$= 2$

Solution :-

$F(x_i) = x_i^3 + 2x_i^2 + 3$

$F'(x_i) = 3x_i^2 + 4x_i$

$F''(x_i) = 6x_i + 4$

$F(x_i) = 4^3 + 2(4)^2 + 3 = 99$

$F'(x_i) = 3(4)^2 + 4(4) = 64$

$F''(x_i) = 6 * 4 + 4 = 28$

① Zero - order approx :-

$F(x_{i+1}) = F(x_i)$

$= 99$

$|\varepsilon_t| = \left|\dfrac{270 - 99}{270}\right| * 100 = 63.34\%$

② First - order approx :-

$F(x_{i+1}) = F(x_i) + F'(x_i)\,h$

$F(6) = 99 + 64(2) = 227$

$|\varepsilon_t| = \left|\dfrac{270 - 227}{270}\right| * 100$

$= 15.93\%$

③ Second order approx. :-

$F(x_{i+1}) = F(x_i) + F'(x_i)\,h + \dfrac{F''(x_i)\,h^2}{2}$

$= 99 + 64 * 2 + \dfrac{28 * 2^2}{2} = 283$

$$\varepsilon_t = \left| \frac{270 - 283}{270} \right| * 100$$

$$= 4.82 \%$$

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4.11 The Maclaurin series expansion for cos x is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Starting with the simplest version, cos $x = 1$, add terms one at a time to estimate $\cos(\pi/4)$. After each new term is added, compute the true and approximate percent relative errors. Use your pocket calculator or MATLAB to determine the true value. Add terms until the absolute value of the approximate error estimate falls below an error criterion conforming to two significant figures. $n = 2$

\# $\varepsilon_s = 0.5 \times 10^{2-n} = 0.5 \times 10^{2-2} = \boxed{0.5\ \%}$

\# $\cos \pi/4 \approx 0.7071$ } true value

⟹ zero order :–

$$\cos(\pi/4) = 1$$

$$\varepsilon_t = \left| \frac{0.7071 - 1}{0.7071} \right| \times 100 = 41.42\ \%$$

⟹ First order :–

$$\cos(\pi/4) = 1 - \frac{(\pi/4)^2}{2} = 0.691575$$

$$\varepsilon_t = \left| \frac{0.7071 - 0.691575}{0.7071} \right| \times 100 = 2.19\ \%$$

$$\varepsilon_a = \left| \frac{0.691575 - 1}{0.691575} \right| \times 100 = 44.6\ \%$$

⟹ Second order :–

$$\cos(\pi/4) = 1 - \frac{(\pi/4)^2}{2} + \frac{(\pi/4)^4}{4!}$$

$$= 0.707429$$

$$\varepsilon_t = \left| \qquad \right| \times 100 = 0.456\ \%$$

$$\varepsilon_a = \left| \qquad \right| \times 100 = 2.24\ \%$$

⟹ third order :–

$\cos \pi/4 =$ ——— ⟹ ✓

$\varepsilon_t =$ ✓

$\varepsilon_a =$ ✓

4.10 The following infinite series can be used to approximate e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

(a) Prove that this Maclaurin series expansion is a special case of the Taylor series expansion (Eq. 4.13) with $x_i = 0$ and $h = x$.

(b) Use the Taylor series to estimate $f(x) = e^{-x}$ at $x_{i+1} = 1$ for $x_i = 0.25$. Employ the zero-, first-, second-, and third-order versions and compute the $|\varepsilon_t|$ for each case.

$$\boxed{F(x) = e^{-x}}$$

$x_{i+1} = 1$

$x_i = 0.25$

$h = x_{i+1} - x_i = 1 - 0.25 = 0.75$

$F(x_i) = e^{-x_i}$

$F'(x_i) = -e^{-x_i}$

$F''(x_i) = +e^{-x_i}$

$F'''(x_i) = -e^{-x_i}$

$F(1) = e^{-1} = 0.3678$ } true value

* Zero order approx :—

$F(x_{i+1}) = F(x_i)$

$F(1) = e^{-x_i} = e^{-0.25}$

$= 0.778801$

$\varepsilon_t = \left|\frac{0.3678 - 0.778801}{0.3678}\right| * 100 = 111.7\%$

* 1st order approx :—

$F(x_{i+1}) = F(x_i) + F'(x_i)\,h$

$= e^{-x_i} - e^{-x_i}\,h$

$F(1) = e^{-0.25} - e^{-0.25}(0.75)$

$= 0.1947$

$\varepsilon_t = \left| \quad \right| * 100 = 47.1\%$

* 2nd order approx :—

$$F(x_{i+1}) = F(x_i) + F'(x_i)h + \frac{F''(x_i)h^2}{2!}$$

$$= 0.1947 + \frac{e^{-0.25} * 0.75}{2!}$$

$$= 0.413738$$

$$\varepsilon_t = \left|\frac{0.3678 - 0.4137}{0.3678}\right| * 100 = 12.5\%$$

* 3rd order approx :—

$$F(1) = F(x_i) + F'(x_i)h + \frac{F''(x_i)h^2}{2!} + \frac{F'''(x_i)h^3}{3!}$$

$$= 0.413738 - \frac{e^{-0.25} * 0.75^3}{3*2*1}$$

$$= 0.358978$$

$$\varepsilon_t = \left|\frac{0.3678 - 0.3589}{0.3678}\right| * 100$$

$$= 2.42\%$$

4.13 Use zero- through third-order Taylor series expansions to predict $f(3)$ for

$$f(x) = 25x^3 - 6x^2 + 7x - 88$$

using a base point at $x = 1$. Compute the true percent relative error ε_t for each approximation.

$$\left.\begin{aligned} x_{i+1} &= 3 \\ x_i &= 1 \end{aligned}\right\} \quad h = x_{i+1} - x_i = 3 - 1 = 2$$

true value

$$F(3) = 25(3)^3 - 6(3)^2 + 7(3) - 88 = 554$$

Taylor Series :-

$$F(x_i) = 25x^3 - 6x^2 + 7x - 88$$

$$F'(x_i) = 75x^2 - 12x + 7$$

$$F''(x_i) = 150x - 12$$

$$F'''(x_i) = 150$$

$$F(x_i) = F(1) = 25(1)^3 - 6(1)^2 + 7(1) - 88$$

$$F(x_i) = -62$$

$$F'(x_i) = 75(1)^2 - 12(1) + 7 = 70$$

$$F''(x_i) = F''(1) = 150(1) - 12 = 138$$

$$F'''(x_i) = F'''(1) = 150$$

Zero order :-

$$F(x_{i+1}) = F(x_i) = -62$$

$$\varepsilon_t = \left| \frac{554 - (-62)}{554} \right| * 100 = 111.19\ \%$$

1st order :-

$$F(x_{i+1}) = F(x_i) + F'(x_i)\, h$$

$$= -62 + 70 * 2$$

$$= 78$$

$$\varepsilon_t = 85.92\ \%$$

2nd-order :-

$$F(x_{i+1}) = \boxed{F(x_i) + F'(x_i)\,h} + \frac{F''(x_i)\,h^2}{2!}$$

$$F(3) = 78 + \frac{138 * 2^2}{2}$$

$$= 354$$

$$\varepsilon_t = \left|\frac{554 - 354}{554}\right| * 100$$

$$= 36.1\ \%$$

Third order :-

$$F(3) = F(x_i) + F'(x_i)\,h + \frac{F''(x_i)\,h^2}{2!} + \frac{F'''(x_i)\,h^3}{3!}$$

$$= 354 + \frac{150 * 2^3}{3 * 2 * 1}$$

$$= 554$$

$$\varepsilon_t = \left|\frac{554 - 554}{554}\right| * 100$$

$$= 0\ \%$$

* ERROR
* Taylor Series
* H.W (1)

Homework 1

Numerical Methods

Problem 1

The MacLaurin series expansion for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Add terms one at a time to estimate $\cos(35^o)$. After each new term is added, compute the approximate relative error. Stop when the approximate relative error is below 0.05%. Use four significant figures throughout your calculations.

* $35^\circ = 35 * \pi/180 = \frac{7\pi}{36}$

* $\varepsilon_s = 0.05\ \%$

$\Rightarrow$ Zer order :-

$\cos 35^\circ = 1$

$\Rightarrow$ First order :-

$\cos 35^\circ = 1 - \frac{(7\pi/36)^2}{2} = 0.8134$

$\varepsilon_a = \left|\frac{0.8134 - 1}{0.8134}\right| * 100 = 22.94\ \% > 0.05\ \%$

Second order :-

$\cos 35^\circ = 1 - \frac{(7\pi/36)^2}{2} + \frac{(7\pi/36)^4}{4!}$

$= 0.8192$

$\varepsilon_a = \left|\frac{0.8192 - 0.8134}{0.8192}\right| * 100 = 0.708\ \%$

$> 0.05\ \%$

Third order :-

$\cos 35^\circ = 1 - \frac{(7\pi/36)^2}{2} + \frac{(7\pi/36)^4}{4!} - \frac{(7\pi/36)^6}{6!}$

$= 0.8192$

$\varepsilon_a = \left|\frac{0.8192 - 0.8192}{0.81915}\right| * 100$

$\approx 0 < 0.05\ \%$

Stop

$\cos 35^\circ = 0.8192$

Problem 2

Consider the function $f(x) = 1 + xe^x$. Noting that $f(0) = 1$, use Taylor series to approximate the value of $f(1)$ with an approximate relative error is below 1%. Use four significant figures throughout your calculations.

Base Point

$$x_i = 0 \implies F(x_i) = 1$$

$$x_{i+1} = 1$$

$$h = x_{i+1} - x_i = 1$$

$$F(x_{i+1}) = F(x_i) + F'(x_i)\,h + \frac{F''(x_i)\,h^2}{2!} + \frac{F'''(x_i)\,h^3}{3!} + \cdots$$

Zero order :-

$$F(x_{i+1}) = F(x_i) = 1$$

1st order :-

$$F(x_{i+1}) = F(x_i) + F'(x_i)\,h$$

$$F(x) = 1 + xe^x$$

$$F'(x) = e^x + xe^x$$

$$F'(x_i) = F'(0) = 1 + 0 = 1$$

$$F(x_{i+1}) = 1 + 1 * 1 = 2$$

$$\varepsilon_a = \left| \frac{2 - 1}{2} \right| * 100 = 50\,\% > 1\%$$

2nd - order :-

$$F(x_{i+1}) = F(x_i) + F'(x_i)\,h + \frac{F''(x_i)\,h^2}{2!}$$

$$F''(x) = e^x + e^x + xe^x$$

$$= 2e^x + xe^x$$

$$F''(x_i) = F''(0) = 2 + 0 = 2$$

$$F(x_{i+1}) = 2 + \frac{\cancel{2} * 1^2}{\cancel{2} * 1} = 3$$

$$\varepsilon_a = \left| \frac{3 - 2}{3} \right| * 100 = 33.33\% > 1\%$$

3rd order

$$F(x_{i+1}) = F(x_i) + F'(x_i)h + \frac{F''(x_i)h^2}{2!} + \frac{F'''(x_i)h^3}{3!}$$

$$F''(x) = 2e^x + xe^x$$

$$F'''(x) = 2e^x + e^x + xe^x$$

$$= 3e^x + xe^x$$

$$F'''(x_i) = F'''(0) = 3 + 0 = 3$$

$$F(x_{i+1}) = 3 + \frac{\cancel{3} * 1^3}{\cancel{3} * 2} = 3.5$$

$$\varepsilon_a = \left| \frac{3.5 - 3}{3.5} \right| * 100 = 14.28\% > 1\%$$

4th order

$$F(x_{i+1}) = F(x_i) + F'(x_i)h + \frac{F''(x_i)h^2}{2!} + \frac{F'''(x_i)h^3}{3!} + \frac{F''''(x_i)h^4}{4!}$$

$$F''''(x) = 4e^x + xe^x$$

$$F''''(0) = 4 + 0 = 4$$

$$F(x_{i+1}) = 3.5 + \frac{4 * 1^4}{4!} = 3.667$$

$$\varepsilon_a = \left| \frac{3.67 - 3.5}{3.67} \right| * 100 = 4.63\%$$

$> 1\%$

5th order :-

$$F'''''(x) = 5e^x + xe^x$$

$$F'''''(0) = 5$$

$$F(x_{i+1}) = 3.667 + \frac{5 * 1^5}{5!} = 3.709$$

$$\varepsilon_a = \left| \frac{3.709 - 3.667}{3.709} \right| * 100 = 1.132\%$$

$> 1\%$

6th - order

$$F^{(6)}(x) = 6e^{x} + xe^{x}$$

$$F^{(6)}(0) = 6$$

$$F(x_{i+1}) = 3.709 + \frac{6 \times 1^{6}}{6!}$$

$$= 3.717$$

$$\varepsilon_a = \left| \frac{3.717 - 3.709}{3.717} \right| \times 100 = 0.2152 < 1\% \quad \text{Stop}$$

$$F(x_{i+1}) = F(1) = 3.717 \quad @ \quad 6\text{th-order}$$

true value $\Rightarrow F(1) = 1 + (1)e^{1} = 3.718$