Numerical Methods

chapter ①

true error (E_t) = true value − approximation

true percent error = $\dfrac{E_t}{\text{true value}} * 100$ %   ⊕ ⊖

if we cannot get true value :—

we use approximation value

$$\begin{matrix}\oplus \\ \ominus\end{matrix}\Big\} \ \varepsilon_a = \frac{\text{current approx} - \text{perrious approx}}{\text{current approx}} \times 100$$

$|\varepsilon_a| < \varepsilon_s$ $\boxed{\text{Stop}}$

$\Downarrow$

tolerance

$\varepsilon_s = 0.5 \times 10^{2-n}$ ع   ⓝ significant Figures

Example :

$e^{0.5} = 1.6487$

Maclaurin series expansion

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

- Solve for $e^{(0.5)}$ until the approx. error estimate $|\varepsilon_a|$ falls below a prespecified error criterion ε_s conforming to three significant figures:

Solution

$$\varepsilon_s = (0.5 * 10^{2-n})\,\% = (0.5 * 10^{2-3}) = 0.05\,\%$$

$$* \quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \text{-------} \frac{x^n}{n!}$$

①st term :-

$$e^x = 1 \quad \Rightarrow e^{0.5} = 1$$

$$\varepsilon_t = \left| \frac{1.6487 - 1}{1.6487} \right| * 100 = 39.3\,\%$$

④ $1.27 > \varepsilon_s$ ×

⑤ $0.158 > \varepsilon_s$ ×

⑥ $0.0158 < \varepsilon_s$ ✓

$e^{0.5} = 1.64869$

2nd term :-

$$e^x = 1 + x \qquad e^{0.5} = 1 + 0.5 = 1.5$$

$$\varepsilon_t = \left| \frac{1.6487 - 1.5}{1.6487} \right| * 100 = 9.02\,\%$$

$$\varepsilon_a = \left| \frac{1.5 - 1}{1.5} \right| * 100 = 33.3\,\% > \boxed{\varepsilon_s}$$

3rd term :-

$$e^x = 1 + x + \frac{x^2}{2} \Rightarrow e^{0.5} = 1 + 0.5 + \frac{0.5^2}{2}$$

$$= 1.625$$

$$\varepsilon_t = \frac{1.6487 - 1.625}{1.6487} * 100$$

$$= 1.44\,\%$$

$$\varepsilon_a = \frac{1.625 - 1.5}{1.625} * 100$$

$$= 7.69\,\% > \varepsilon_s$$

The Taylor Series :-

→ to predict a function value in approximate fashion.

$$\underset{\text{Required}}{F(x_{i+1})} = \overset{\text{base point}}{F(x_i)} + F'(x_i)\overbrace{(x_{i+1} - x_i)}^{h \Rightarrow \text{step}} + \frac{F''(x_i)}{2!}(h)^2 + \frac{F'''(x_i)}{3!}(h)^3 + \cdots + \boxed{R_n}$$

Zero-order approximation :- 1st-term

$$F(x_{i+1}) = F(x_i)$$

1st order approximation :- 2-term

$$F(x_{i+1}) = F(x_i) + F'(x_i)\, h$$

2nd order approximation :-

$$F(x_{i+1}) = F(x_i) + F'(x_i)\, h + \frac{F''(x_i)}{2!} h^2$$

$$\frac{d}{dx}\left(e^{-x}\right) = (-1)\,e^{-x}$$

4.10 The following infinite series can be used to approximate e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

(a) Prove that this Maclaurin series expansion is a special case of the Taylor series expansion (Eq. 4.13) with $x_i = 0$ and $h = x$.

(b) Use the Taylor series to estimate $f(x) = e^{-x}$ at $x_{i+1} = 1$ for $x_i = 0.25$. Employ the zero-, first-, second-, and third-order versions and compute the $|\varepsilon_t|$ for each case.

Solution :-

$$\cancelto{x}{h} = x_{i+1} - \cancelto{0}{x_i}$$

ⓐ $F(x) = e^x$

$x_i = 0$ $\qquad h = x$ $\qquad x_{i+1} = x$

$F(x_{i+1}) = F(x_i) + F'(x_i)\,h + \frac{F''(x_i)}{2!}h^2 + \cdots\cdots$

① $\Longrightarrow F(x) = F(0) + F'(0)\,x + \frac{F''(0)}{2!}x^2 + \cdots\cdots$

$F(x) = e^x$ $\qquad F' = e^x$

$F(0) = 1$ $\qquad F'(0) = e^0 = 1$ $\qquad F''(0) = 1$

in Eqⁿ ①

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\cdots \frac{x^n}{n!}$

Ⓑ $x_{i+1} = 1$ $\qquad x_i = 0.25$ $\qquad h = 1 - 0.25 = 0.75$

$$F(x_{i+1}) = F(x_i) + F'(x_i)\,h + \frac{F''(x_i)h^2}{2!} + \frac{F'''(x_i)\,h^3}{3!}$$

ⓑ $x_{i+1} = 1$ $x_i = 0.25$ $h = 1 - 0.25 = 0.75$

$$F(x_{i+1}) = F(\underline{x_i}) + \boxed{F'(x_i)\, h} + \frac{F''(x_i)h^2}{2!} + \frac{F'''(x_i)\, h^3}{3!}$$

true value

$$F(1) = e^{(-1)} = 0.3678$$

$\boxed{F(x) = e^{-x}}$

$F(x_i) = e^{-x_i}$

$F'(x_i) = -e^{-x_i}$

$F''(x_i) = +e^{-x_i}$

$$F(x_{i+1}) = e^{-x_i} \boxed{-e^{-x_i} h} + e^{-x_i}\frac{h^2}{2} - e^{-x_i}\frac{h^3}{3!} \Leftarrow$$

Zero order approx :-

$$F(1) = e^{-0.25} = \boxed{0.778801}$$

$$\varepsilon_t = \left|\frac{0.3678 - 0.778801}{0.3678}\right| * 100 = 111.7\ \%$$

1st order approx :-

$$F(1) = 0.778801 - e^{-0.25}(0.75) = 0.1947$$

$$\varepsilon_t = \left|\frac{0.3678 - 0.1947}{0.3678}\right| * 100 = 47.1\ \%$$

2nd order approx

$$F(1) = 0.778801 - (0.778801)(0.75) + 0.778801 * \frac{0.75^2}{2} = 0.413738$$

$$\varepsilon_t = \left|\frac{0.3678 - 0.413738}{0.3678}\right| * 100 = 12.5\ \%$$

3rd order approx :-

$$F(1) = 0.778801 - 0.778801(0.75) + \frac{0.778801\,(0.75)^2}{2!} - 0.778801 * \frac{0.75^3}{3!} = 0.358978$$

$$\varepsilon_t = \left|\frac{0.3678 - 0.358}{0.3678}\right| * 100 = 2.42\ \%$$

4.11 The Maclaurin series expansion for cos x is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Starting with the simplest version, $\cos x = 1$, add terms one at a time to estimate $\cos(\pi/4)$. After each new term is added, compute the true and approximate percent relative errors. Use your pocket calculator or MATLAB to determine the true value. Add terms until the absolute value of the approximate error estimate falls below an error criterion conforming to two significant figures.

$$\varepsilon_s = 0.5 * 10^{2-n} = 0.5 * 10^{2-2} = \boxed{0.5\ \%}$$

true value $\quad \cos(\pi/4) \cong 0.7071$

Zero order :-

$$\cos(\pi/4) = 1 \quad \Big\} \quad \varepsilon_t = \left| \frac{0.7071 - 1}{0.7071} \right| * 100$$

$$= 41.42\ \%$$

First order :-

$$\cos(\pi/4) = 1 - \frac{(\pi/4)^2}{2} = 0.691575$$

$$\varepsilon_t = \left| \frac{0.7071 - 0.691575}{0.7071} \right| * 100 = 2.19\ \%$$

$$\varepsilon_a = \left| \frac{0.691575 - 1}{0.691575} \right| * 100 = \boxed{44.6}\ \%$$

Second order :-

$$\cos \pi/4 = \underbrace{1 - \frac{(\pi/4)^2}{2}} + \boxed{\frac{(\pi/4)^4}{4!}} = 0.691575 + \boxed{}$$

$$= \boxed{0.707429} \leftarrow$$

$$\varepsilon_t = \left| \frac{0.7071 - 0.7074}{0.7071} \right| * 100 = 0.456\ \%$$

$$\varepsilon_a = \left| \frac{0.707429 - 0.691575}{0.707429} \right| * 100 = \boxed{2.24\ \%}$$

Third order :-

$$\cos\left(\pi/4\right) = 0.707429 - \frac{\left(\pi/4\right)^6}{6!} = 0.707103$$

$$\varepsilon_t = \left| \frac{0.707107 - 0.707103}{0.707107} \right| * 100 = 0.0005\ \%$$

$$\varepsilon_a = \left| \frac{0.\underline{70}7103 - \boxed{0.707429}}{0.\underline{707}103} \right| * 100 = 0.046\ \% \quad < 0.5\ \%$$

terminate ✓

4.13 Use zero- through third-order Taylor series expansions to predict $f(3)$ for

$$f(x) = 25x^3 - 6x^2 + 7x - 88$$

using a base point at $x = 1$. Compute the true percent relative error ε_t for each approximation.

$x_{i+1} = 3$ $h = 2$

$x_i = 1$

\# true value $F(3) = 25(3)^3 - 6(3)^2 + 7(3) - 88$

$= 554$

\# $F(x_{i+1}) = F(x_i) + F'(x_i)\,h + F''(x_i)\frac{h^2}{2} + F'''(x_i)\frac{h^3}{3!}$

$= F(1) + F'(1)(2) + F''(1)\frac{2^2}{2} + F'''(1)\frac{2^3}{3!}$

$F(1) = 25(1)^3 - 6(1)^2 + 7(1) - 88 = -62$

$F'(x) = 75x^2 - 12x + 7$

$F'(1) = 75(1)^2 - 12(1) + 7 = 70$

$F''(x) = 150x - 12$

$F''(1) = 150(1) - 12 = 138$

$F'''(x) = \boxed{150}$

\# zero order

$F(3) = F(1) = -62$

$\varepsilon_t = \left| \frac{554 - (-62)}{554} \right| \times 100$

$= 111.19\ \%$

\# 1st order :-

$F(3) = F(1) + F'(1)(2) = -62 + (70)(2) = 78$

$\varepsilon_t = 85.92\%$

\# 2nd order

$F(3) = F(1) + F'(1)(2) + F''(1)\frac{2^2}{2!}$

$$F(3) = -62 + 70(2) + \frac{138}{2}(2)^2 = \boxed{354}$$

$$\varepsilon_t = \left| \frac{554 - 354}{554} \right| * 100$$

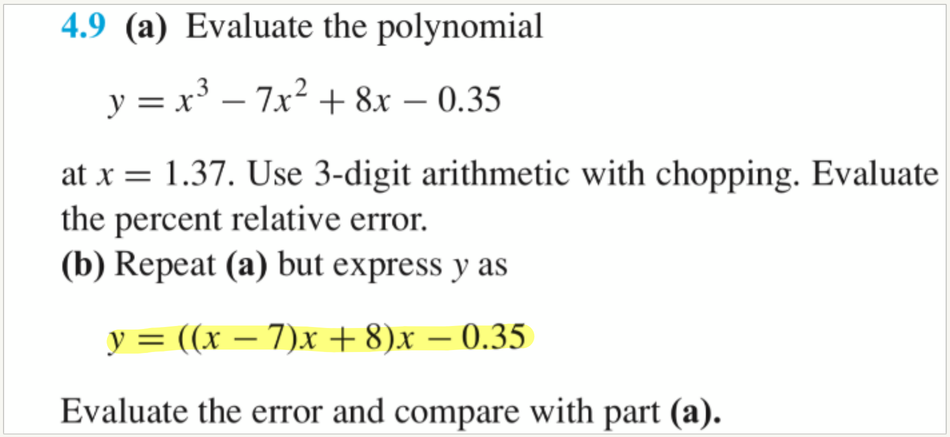
$$= 36.1\ \%$$

$\boxed{\text{Third order}}$

$$F(3) = 354 + \frac{150(2)^3}{3*2*1} = 554$$

$$\varepsilon_t = \left| \frac{554 - 554}{554} \right| * 100$$

$$= 0\ \%$$



4.9 **(a)** Evaluate the polynomial

$$y = x^3 - 7x^2 + 8x - 0.35$$

at $x = 1.37$. Use 3-digit arithmetic with chopping. Evaluate the percent relative error.
(b) Repeat **(a)** but express y as

$$y = ((x - 7)x + 8)x - 0.35$$

Evaluate the error and compare with part **(a)**.

true value

$$y = (1.37)^3 - 7(1.37)^2 + 8(1.37) - 0.35$$

$$= 0.043053$$

ⓐ using 3-digits with chopping

$$(1.37)^3 \longrightarrow 2.571353 \longrightarrow 2.57$$

$$-7(1.37)^2 \longrightarrow -7(1.87) \longrightarrow -13.0$$

$$8(1.37) \longrightarrow 10.96 \longrightarrow 10.9$$

$$-0.35$$

$$+0.12$$

$$\varepsilon_t = \left|\frac{0.043053 - 0.12}{0.043053}\right| \times 100 = \boxed{178.7\%}$$

ⓑ

$$y = ((x-7)x + 8)x - 0.35$$

$$= ((1.37-7)1.37 + 8)1.37 - 0.35$$

$$= ((-5.63)(1.37) + 8)1.37 - 0.35$$

$$= (-7.71 + 8)1.37 - 0.35$$

$$= (0.29)1.37 - 0.35$$

$$= 0.397 - 0.35 = 0.047$$

$$\varepsilon_t = \left|\frac{0.043053 - 0.047}{0.043053}\right| \times 100 = \boxed{9.2\%}$$

The second is better it tends to minimize round-off error

Example

Use Taylor Series to approximate f(x) = cos(x) at $x_i = \pi/4$ on the basis of value of $x_{i+1} = \pi/3$

The correct value $f(\pi/3) = 0.5$

Zero-order approximation is f(π/3) ≅ f(π/4) = 0.707106781

& relative error

$$\boldsymbol{\varepsilon_t = \frac{0.5 - 0.707106781}{0.5} \quad 100\% = 41.4\%}$$

First-order approximation is

$$f(\pi/3) \cong \cos(\pi/4) + (-\sin(\pi/4)\,(\pi/3 - \pi/4) = 0.521986659$$

$$\varepsilon_t = -4.41\%$$

Example

Use Taylor Series to approximate $f(x) = Cos\,x$ at $x_i = \pi/4$ on the basis of value of $x_{i+1} = \pi/3$

Order n	$f^{(n)}(x)$	$f(\pi/3)$	ε_t
0	Cos x	0.707106781	-41.4
1	-Sin x	0.521986659	-4.4
2	-Cos x	0.497754491	0.449
3	Sin x	0.499869147	0.0262
4	Cos x	0.500007551	-0.00151
5	-Sin x	0.500000304	-0.0000608
6	-Cos x	0.499999988	0.00000244