Roots of Equations

Chapter 5

* $ax^2 + bx + c = 0$ $\quad\Rightarrow\quad$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

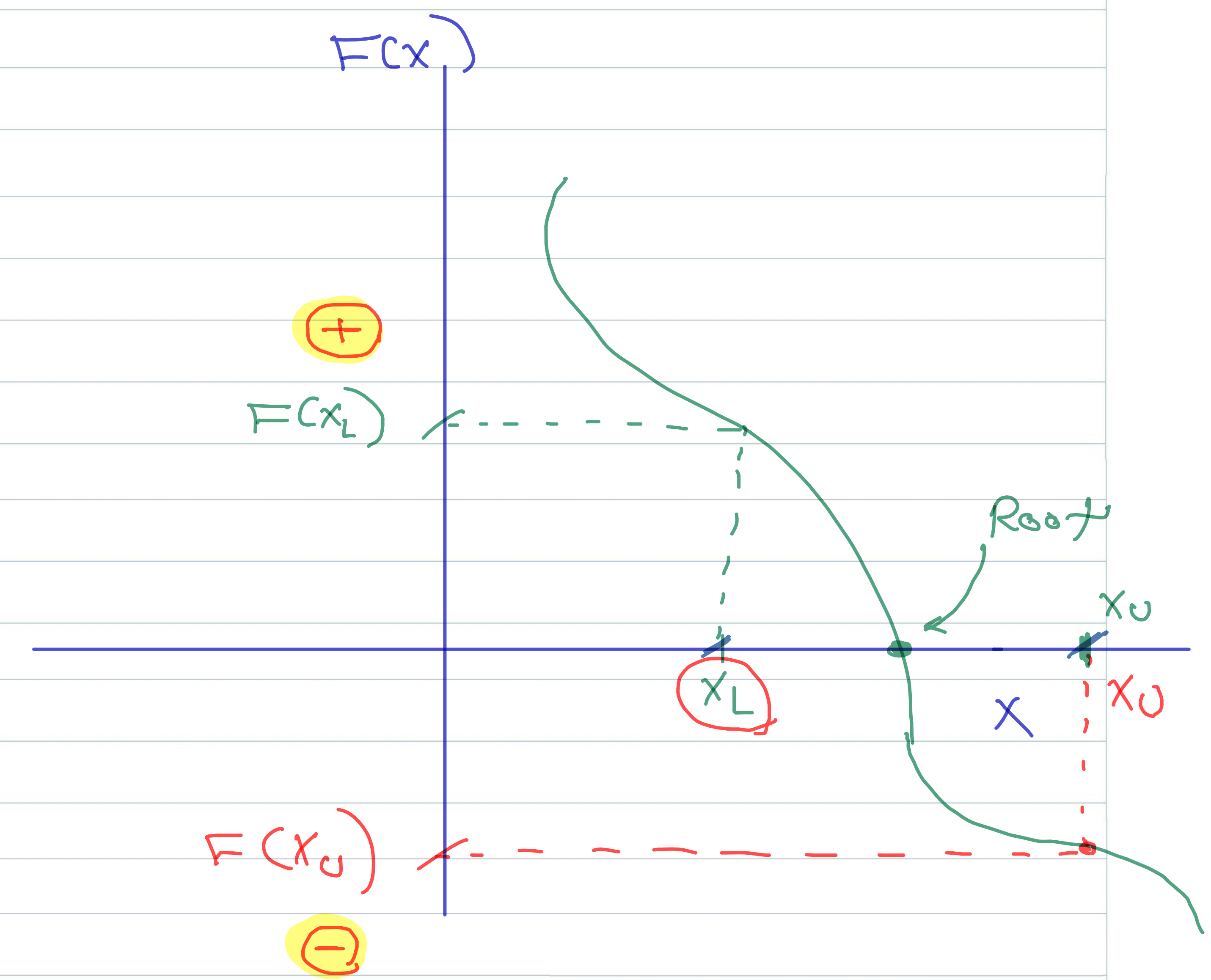
* non-linear :-

$F(x) = \sin x + x = 0$ $\quad\Rightarrow\quad$ $x = ???$

- graphical
- Bracketing
  - Bisection
  - False Method
- open Method
  - Newton Raphson
  - Secant

graphical

$F(x) = \sin x + x$



Bracketing Methods

(Or, two point methods for finding roots)

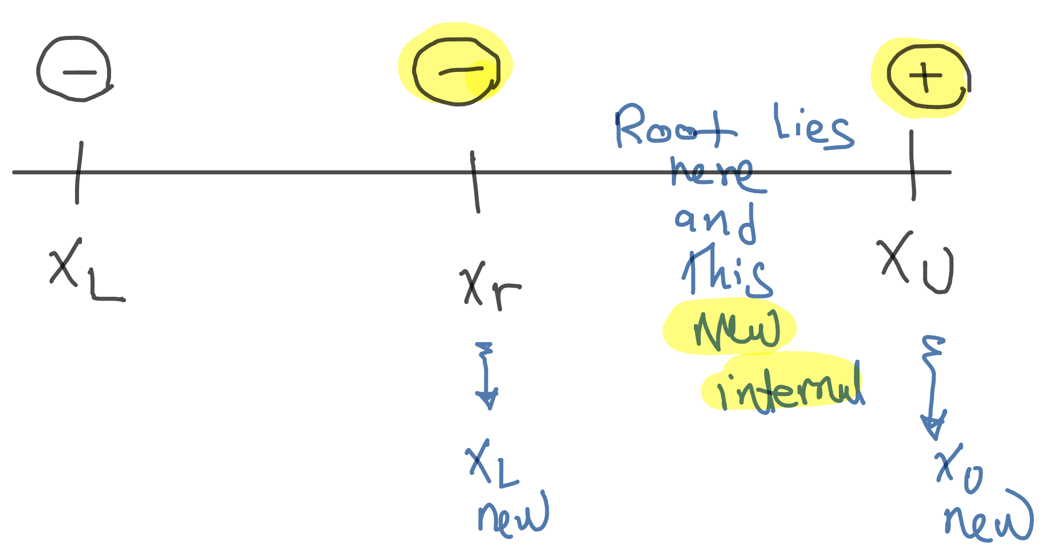
The Bisection Method

1) Pick $\left.\begin{matrix} x_L \\ x_U \end{matrix}\right\}$ $\quad \underset{\ominus}{F(x_L)} * \underset{\oplus}{F(x_U)} < 0$

2) $x_r = \dfrac{x_L + x_U}{2}$

$F(x_r) = \ominus$

3)



$\varepsilon_a = \left| \dfrac{\quad\quad}{\quad\quad} \right| * 100$

4) new interval $\left.\begin{matrix} \ominus \\ \text{New } x_L \\ \text{New } x_U \\ \oplus \end{matrix}\right\}$ $\quad x_r = \dfrac{x_L + x_U}{2}$

$\downarrow$

$F(x_r)$

$\downarrow$

New interval

$\downarrow$

ε_a

ε_t

5) Continue until $\varepsilon_a < \varepsilon_s$ otherwise repeat

Example statement :-

Find positive root of the following eqn

$$x^3 - 4x - 9 = 0$$

Using ① Bisection method

The prespecified error is < 2%

① Pick $x_L = ?$

$x_U = ?$

$$F(x) = x^3 - 4x - 9$$

$F(0) = -9 < 0$ -ve

$F(1) = (1)^3 - 4(1) - 9 = -12 < 0$ -ve

$F(2) = (2)^3 - 4(2) - 9 = -9 < 0$ -ve

$F(3) = (3)^3 - 4(3) - 9 = 6 > 0$ +ve

$x_L = 2$

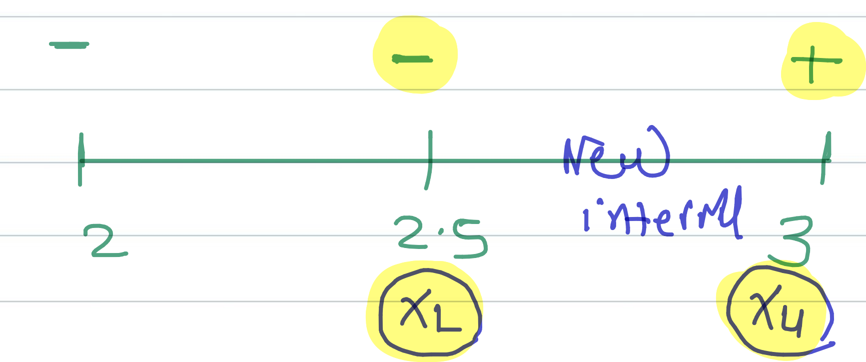
$x_U = 3$ } Root Lies here

Iteration ① :-

* $x_r = \dfrac{x_L + x_U}{2} = \dfrac{2+3}{2} = 2.5$

$F(x_r) = (2.5)^3 - 4(2.5) - 9$

$= -3.375 < 0$ $\boxed{-ve}$

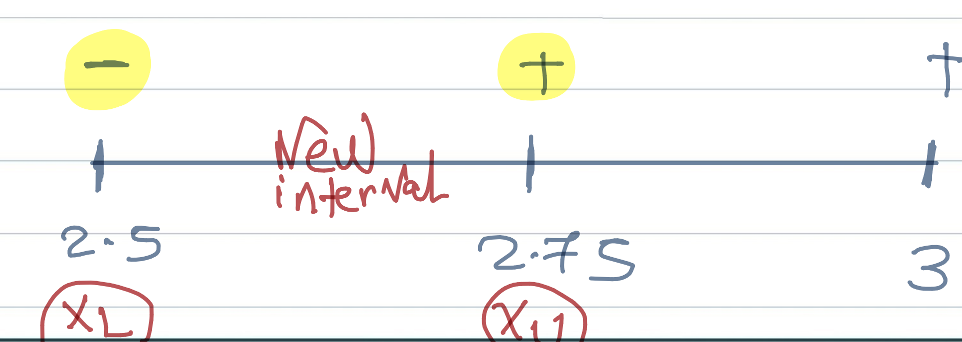
* 

* No ε_a

Iteration ② :-

* $x_r = \dfrac{x_L + x_U}{2} = \dfrac{2.5 + 3}{2} = 2.75$

$F(x_r) = (2.75)^3 - 4(2.75) - 9 = 0.79$ > 0 +ve

* 

$$\varepsilon_a = \left| \frac{X_{r\,current} - X_{r\,prev}}{X_{r\,current}} \right| * 100$$

$$= \left| \frac{2.75 - 2.5}{2.75} \right| * 100 = 9.09\,\% > 2\%$$

$\Downarrow$

Next iteration

Iteration ③ :-

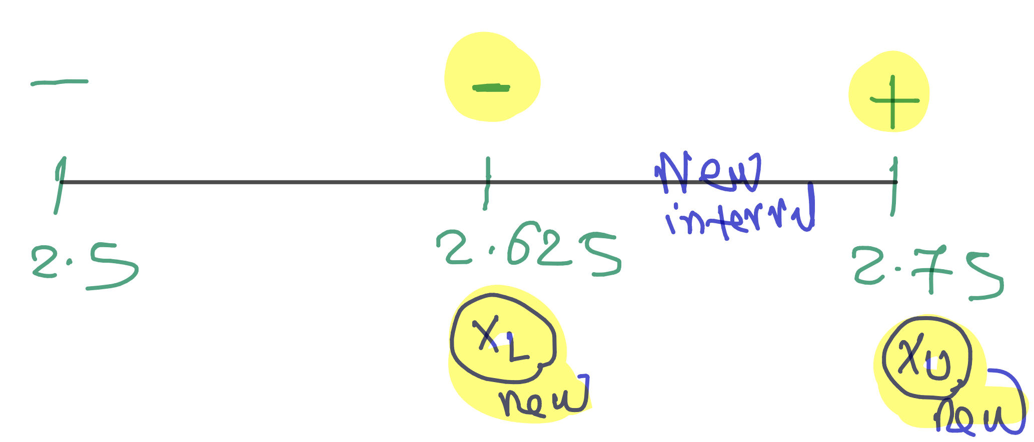
* $X_r = \frac{X_L + X_U}{2} = \frac{2.5 + 2.75}{2} = 2.625$

$$F(X_r) = F(2.625)$$

$$= (2.625)^3 - 4(2.625) - 9$$

$$= -1.412 < 0 \quad -ve$$

*



$$\varepsilon_a = \left| \frac{2.625 - 2.75}{2.625} \right| * 100$$

$$= 4.76\,\% > 2\,\%$$

$\Downarrow$

Next iteration

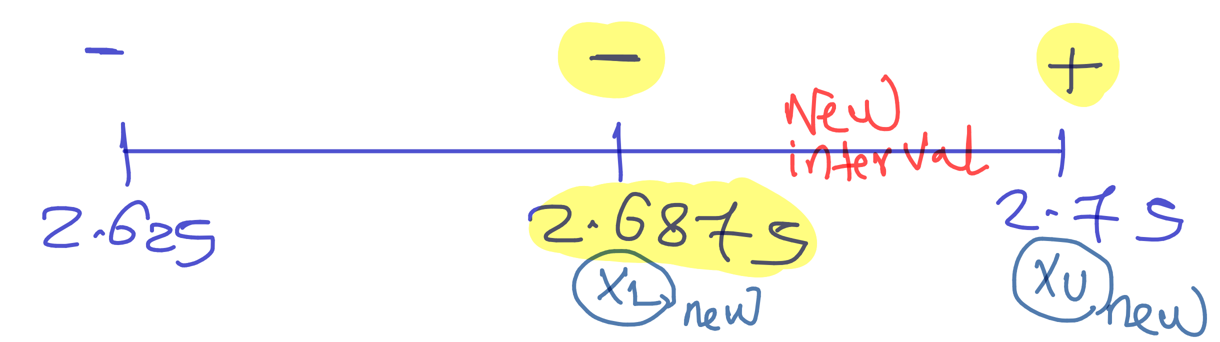
Iteration 4 :-

* $X_r = \frac{X_L + X_U}{2} = \frac{2.625 + 2.75}{2}$

$$= 2.6875$$

$$F(X_r) = (2.6875)^3 - 4(2.6875) - 9$$

$$= -0.339 < 0 \quad -ve$$



$$\varepsilon_a = \left| \frac{2.6875 - 2.75}{2.6875} \right| * 100 = 2.325\,\%$$

$$2.325\,\% > 2\,\%$$

$\Downarrow$

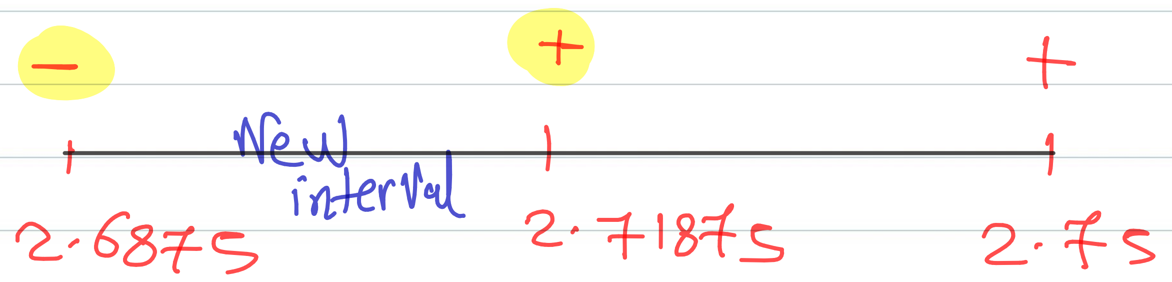
Next iteration

Iteration 5 :—

* $x_r = \dfrac{2.6875 + 2.75}{2} = 2.71875$

$F(2.71875) = 0.22 > 0 \quad +ve$

*



* $\varepsilon_a = \left| \dfrac{2.71875 - 2.6875}{2.7187} \right| * 100$

$= 1.147\,\% < 2\,\%$

Stop

Root is 2.71875

$\varepsilon_a = 1.147\,\% < 2\,\%$

after 5 iteration

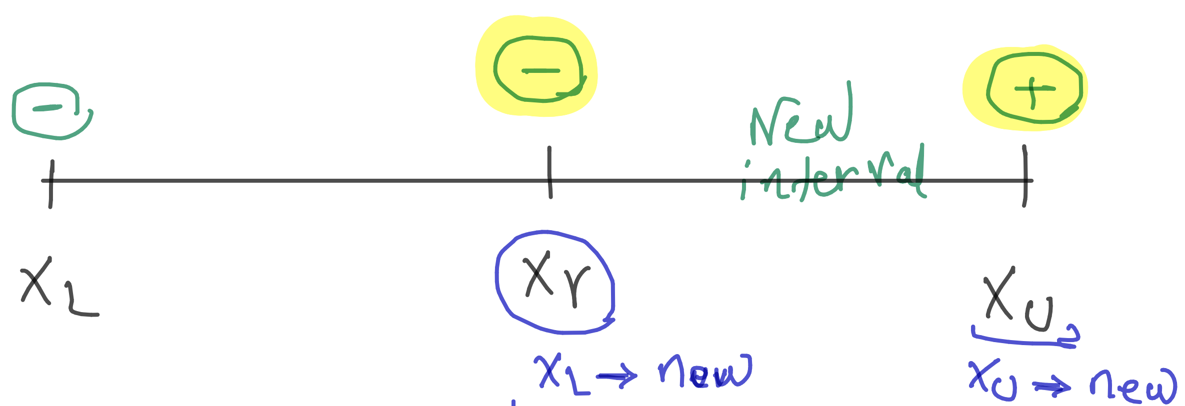
The False-Position Method

1) Pick x_L, x_U $\quad F(x_L)\,F(x_U) < 0$ $\quad (\ominus)\,(\oplus)$

2) $$x_r = x_U - \frac{F(x_U)\,(x_L - x_U)}{F(x_L) - F(x_U)}$$

$$F(x_r) = \ominus$$

3) new interval



$$\varepsilon_a = \left| \qquad \right| * 100$$

4) New iteration $\left.\begin{array}{l} x_L \\ x_U \end{array}\right\}$ x_r, $F(x_r)$ $\Downarrow$ new interval $\downarrow$ ε_a

5) Continue until $\varepsilon_a < \varepsilon_s$

otherwise repeat

False position method

Find positive root of the following equation

$x^3 - 4x - 9 = 0$

Using → False position method

The prespecified Error is 2%

1) Pick $X_L = 2$ $\quad F(2) = -9$

$X_U = 3$ $\quad F(3) = 6$

Iteration ① :-

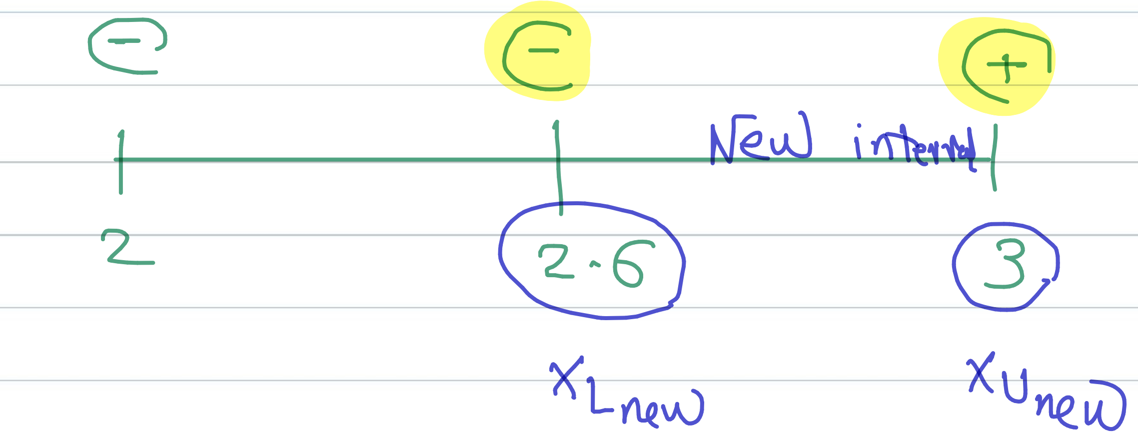
* $$X_r = X_U - \frac{F(X_U)\,(X_L - X_U)}{F(X_L) - F(X_U)}$$

$$= 3 - \frac{6\,(2-3)}{-9-6} = 2.6$$

$$F(X_r) = (2.6)^3 - 4\,(2.6) - 9$$

$$= -1.824$$

*



Iteration ② :-

* $X_L = 2.6$ $\quad F(X_L) = -1.824$

$X_U = 3$ $\quad F(X_U) = 6$

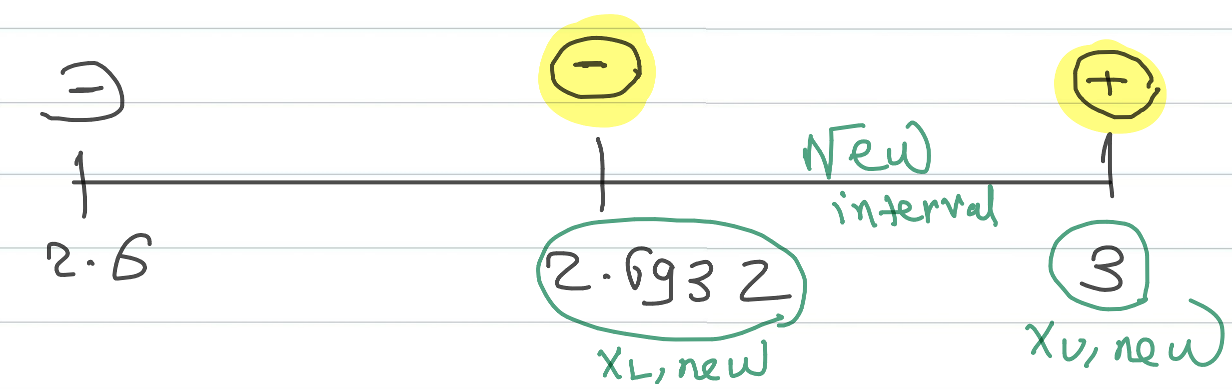
$$X_r = 3 - \frac{6\,(2.6-3)}{(-1.824) - 6}$$

$$= 2.6932$$

$$F(X_r) = F(2.6932) = -0.2381$$

-ve

*



$$\varepsilon_a = \left|\frac{2.6932 - 2.6}{2.6932}\right| * 100 = 3.46\,\%$$

$> 2\,\%$

⇓

Next iteration

Iteration (3) :-

* $x_L = 2.6932$ $\qquad F(x_L) = -0.2381$

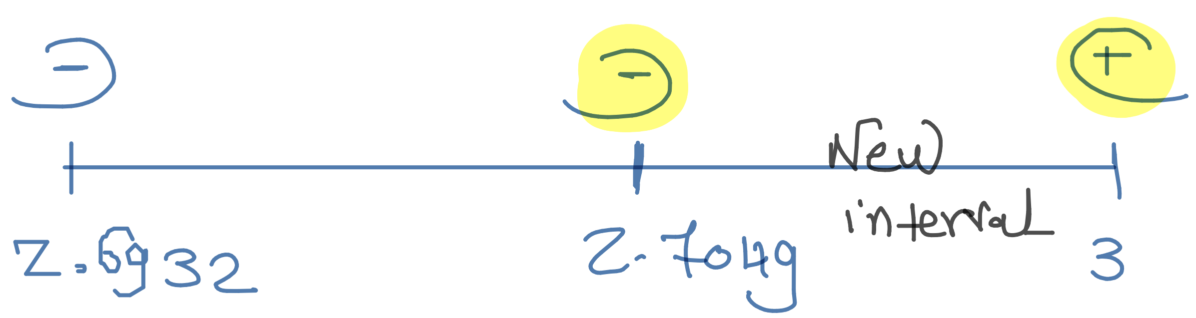
$x_U = 3$ $\qquad F(x_U) = 6$

$$x_r = 3 - \frac{6\,[2.6932 - 3)}{-0.2381 - 6}$$

$$= 2.7049$$

$$F(x_r) = F(2.7049) = -0.02924$$

$\boxed{-ve}$



* $$\varepsilon_a = \left| \frac{2.7049 - 2.6932}{2.7049} \right| * 100$$

$$= 0.4326\,\% < 2\,\%$$

Stop

Root $= 2.7049$ } 3-iteration

$\varepsilon_a = 0.4326 < 2\,\%$

Problem #1

5.1 Use bisection to determine the drag coefficient needed so that an 80-kg bungee jumper has a velocity of 36 m/s after 4 s of free fall. Note: The acceleration of gravity is 9.81 m/s². Start with initial guesses of $x_l = 0.1$ and $x_u = 0.2$ and iterate until the approximate relative error falls below 2%.

M, v, t, g

\# $$F(c_d) = \sqrt{\frac{gM}{c_d}} \tanh\left(t\sqrt{\frac{g c_d}{M}}\right) - v(t)$$

$$F(c_d) = \sqrt{\frac{9.81 * 80}{c_d}} \tanh\left(4\sqrt{\frac{9.81\, c_d}{80}}\right) - 36$$

1

$x_L = 0.1$

$$F(x_L) = \sqrt{\frac{9.81*80}{0.1}} \tanh\left(4\sqrt{\frac{9.81*0.1}{80}}\right) - 36$$

$$= 0.86029$$

$x_U = 0.2$

$$F(x_U) = \sqrt{\frac{9.81*80}{0.2}} \tanh\left(4\sqrt{\frac{9.81*0.2}{80}}\right) - 36$$

$$= -1.19738$$

Iteration 1 :-

* $$x_r = \frac{x_L + x_U}{2} = \frac{0.1 + 0.2}{2} = 0.15$$

$$F(0.15) = \sqrt{\frac{9.81*80}{0.15}} \tanh\left(4\sqrt{\frac{9.81*0.15}{80}}\right) - 36$$

$$= -0.20452$$

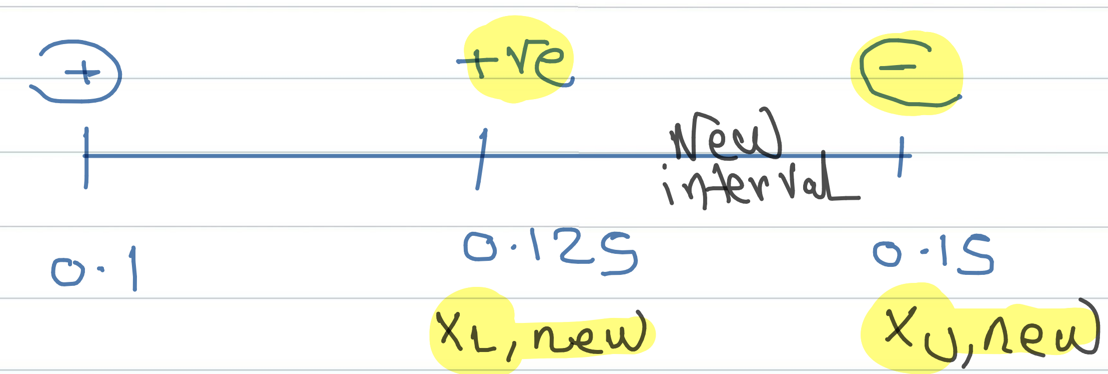
*

Iteration 2 :-

* $$x_r = \frac{x_L + x_U}{2} = \frac{0.1 + 0.15}{2} = 0.125$$

$$F(0.125) = 0.31841 \quad +ve$$

*



$$\varepsilon_a = \left| \frac{0.125 - 0.15}{0.125} \right| * 100 = 20\% > 2\%$$

$\Downarrow$

Next Iteration.

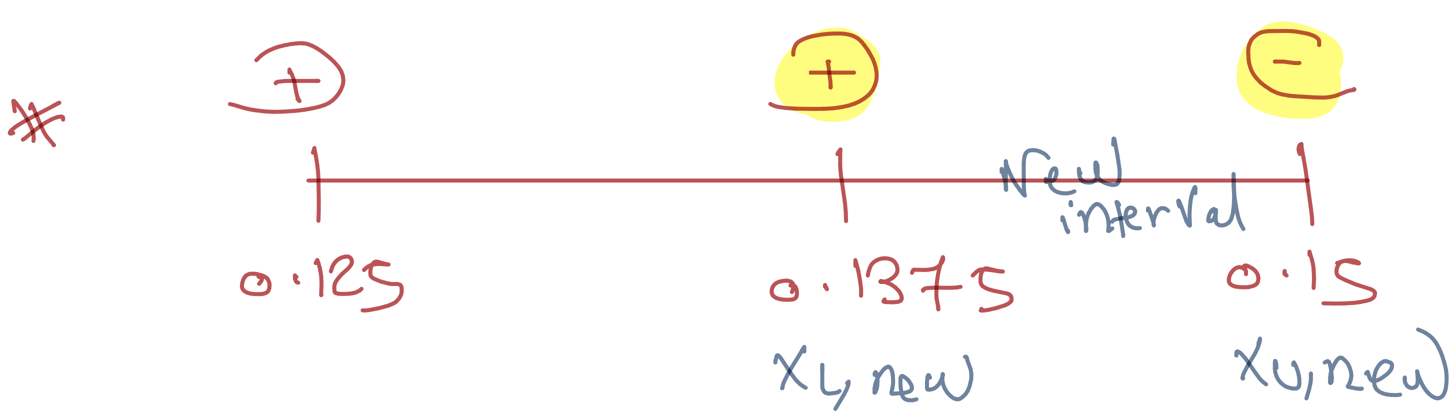
Iteration ③ :-

* $X_L = 0.125$

$X_U = 0.15$

$$X_r = \frac{X_L + X_U}{2} = \frac{0.125 + 0.15}{2} = 0.1375$$

$$F(X_r) = F(0.1375) = 0.05464$$

\# 

* $$\varepsilon_a = \left| \frac{0.1375 - 0.125}{0.1375} \right| * 100$$

$$= 9.09\% > 2\% \Rightarrow \text{Next Iteration}$$

i	x_l	$f(x_l)$	x_u	$f(x_u)$	x_r	$f(x_r)$	$\lvert\varepsilon_a\rvert$
1	0.1	0.86029	0.2	−1.19738	0.15	−0.20452	
2	0.1	0.86029	0.15	−0.20452	0.125	0.31841	20.00%
3	0.125	0.31841	0.15	−0.20452	0.1375	0.05464	9.09%
4	0.1375	0.05464	0.15	−0.20452	0.14375	−0.07551	4.35%
5	0.1375	0.05464	0.14375	−0.07551	0.140625	−0.01058	2.22%
6	0.1375	0.05464	0.140625	−0.01058	0.1390625	0.02199	1.12%

Thus, after six iterations, we obtain a root estimate of **0.1390625** with an approximate error of 1.12%.

Problem #2 Solve the problem 5.1 using false-position method.

$$F(c_d) = \sqrt{\frac{g\,m}{c_d}}\tanh\left(t\sqrt{\frac{g\,c_d}{m}}\right) - v(t)$$

$$F(c_d) = \sqrt{\frac{9.81 * 80}{c_d}}\tanh\left(4\sqrt{\frac{9.81\,c_d}{80}}\right) - 36$$

$x_L = 0.1$

$$F(x_L) = \sqrt{\frac{9.81 * 80}{0.1}}\tanh\left(4\sqrt{\frac{9.81 * 0.1}{80}}\right) - 36$$

$$= \boxed{0.86029}$$

$x_U = 0.2$

$$F(x_U) = \sqrt{\frac{9.81 * 80}{0.2}}\tanh\left(4\sqrt{\frac{9.81 * 0.2}{80}}\right) - 36$$

$$= -1.19738$$

Iteration ① :-

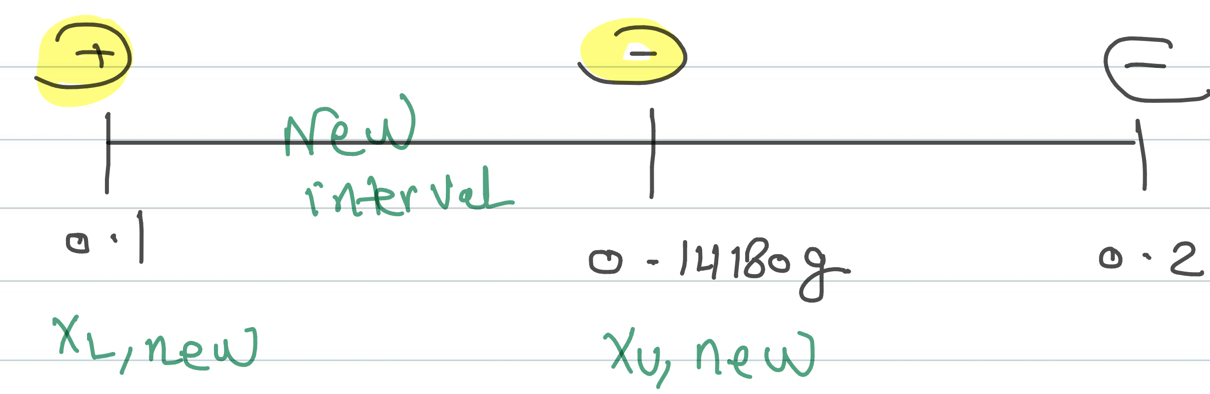
$$* \quad x_r = x_U - \frac{F(x_U)(x_L - x_U)}{F(x_L) - F(x_U)}$$

$$= 0.2 - \frac{-1.19738\,(0.1 - 0.2)}{0.86029 - (-1.19738)}$$

$$= 0.141809$$

$F(x_r) = -0.03521$

*



Iteration ②

* $x_L = 0.1$ $\qquad F(x_L) = 0.86029$

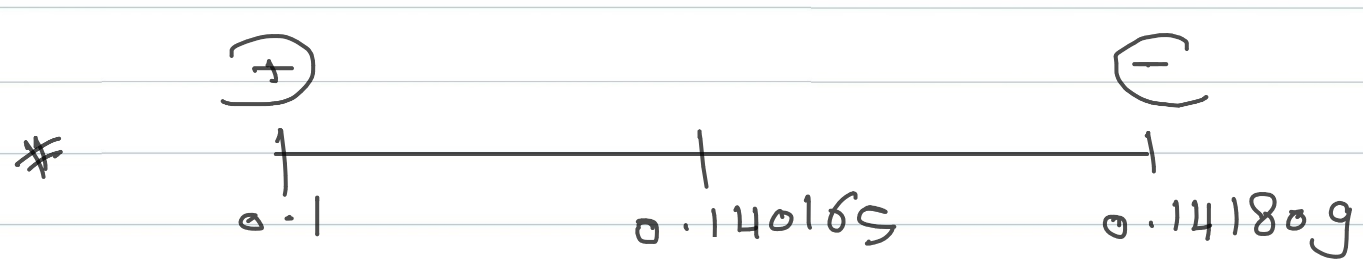
$x_U = 0.141809$ $\qquad F(x_U) = -0.03521$

$$x_r = 0.141809 - \frac{-0.03521\,(0.1 - 0.141809)}{0.86029 + 0.03521}$$

$$= 0.140165$$

$F(x_r) =$ (HW)

*



$$\varepsilon_a = \left| \frac{0.140165 - 0.141809}{0.140165} \right| * 100$$

$$= 1.17\% < 2\% \quad \text{Stop}$$

$$\left.\begin{array}{l} \text{Root} = 0.140165 \\ \varepsilon_a = 1.17\% < 2\% \end{array}\right\} \text{2 - Iterations}$$